MODALITY FOR FREE: NOTES ON ADDING THE TARSKIAN MÖGLICHKEIT TO SUBSTRUCTURAL LOGICS

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Abstract. We briefly examine the modal formulae that can be derived in Multiplicative Additive Linear Logic (**MALL**) and some extensions by using Tarksi's extensional modal operators. We also breifly compare this with a substructural form of the modal logic **K**.

1. Introduction

The Tarskian *möglichkeit* (literally, "possibility" in German) is a modal operator that was introduced by Łukasiewicz (and attributed to Tarski) in [13, §7]. This modal operator is unusual in that it is an *extensional* one, defined in terms of other connectives in Łukasiewicz's many-valued logics:

$$\Diamond A =_{def} \neg A \to A \tag{1}$$

The modal logic that results from this definition is unusual, in part because of the theorems such as:

$$(\Diamond A \wedge \Diamond B) \to \Diamond(A \wedge B) \tag{2}$$

In the case where $B = \neg A$, theorem (2) appears to be paradoxical, if not absurd, and largely because of this, the Tarskian *möglichkeit* has been a footnote in the history of modal logic. Most of the analyses that we are aware of has been for the 3-valued logic, in [12], [8] (but omitted from [9]), [5], and [3], and it is generally critical. An application of the $m > 3$-valued logics to describing m-state systems was suggested in [20], and an application of the infinite-valued logic applied to modelling degrees of believability was suggested by the current author in [17].

However, the infinite-valued logic can be seen as an extension of Affine Logic [4], and many of the modal formulae derivable in the infinite-valued logic are derivable in weaker substructural logics. We give an overview of some of the *formal* properties below, by noting modal rules and formulae in the corresponding logics. We make no claims about the applications of the Tarskian möglichkeit.

2. Multiplicative Additive Linear Logic

The sequent rules for **GMALL**, a calculus for Multiplicative Additive Linear Logic (**MALL**) [6] are given in Figure 1, using notation similar to [19]—in particular, we use $\oplus$ for *par* (multiplicative disjunction) and $\vee$ for *plus* (additive disjunction).

The modal rules (Figure 2) are derived in a straightforward manner. (The corresponding box operator is defined as the dual of diamond operator $\boxdot A =_{def} \neg\Diamond\neg A$.)

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$$\frac{}{0 \Rightarrow}\ \mathsf{L0} \qquad \frac{\Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta, 0}\ \mathsf{R0} \qquad \frac{\Gamma \Rightarrow \Delta}{1, \Gamma \Rightarrow \Delta}\ \mathsf{L1} \qquad \frac{}{\Rightarrow 1}\ \mathsf{R1}$$

$$\frac{}{\bot, \Gamma \Rightarrow \Delta}\ \mathsf{L}\bot \qquad \frac{}{\Gamma \Rightarrow \Delta, \top}\ \mathsf{R}\top$$

$$\frac{\Gamma, A, B \Rightarrow \Delta}{A \otimes B, \Gamma \Rightarrow \Delta}\ \mathsf{L}\otimes \qquad \frac{\Gamma \Rightarrow \Delta, A \quad \Gamma' \Rightarrow \Delta', B}{\Gamma, \Gamma' \Rightarrow \Delta, \Delta', A \otimes B}\ \mathsf{R}\otimes$$

$$\frac{A, \Gamma \Rightarrow \Delta \quad B, \Gamma' \Rightarrow \Delta'}{A \oplus B, \Gamma, \Gamma' \Rightarrow \Delta, \Delta'}\ \mathsf{L}\oplus \qquad \frac{\Gamma \Rightarrow \Delta, A, B}{\Gamma \Rightarrow \Delta, A \oplus B}\ \mathsf{R}\oplus$$

$$\frac{A, \Gamma \Rightarrow \Delta}{A \wedge B, \Gamma \Rightarrow \Delta}\ \mathsf{L}\wedge_1 \qquad \frac{B, \Gamma \Rightarrow \Delta}{A \wedge B, \Gamma \Rightarrow \Delta}\ \mathsf{L}\wedge_2 \qquad \frac{\Gamma \Rightarrow \Delta, A \quad \Gamma \Rightarrow \Delta, B}{\Gamma \Rightarrow \Delta, A \wedge B}\ \mathsf{R}\wedge$$

$$\frac{A, \Gamma \Rightarrow \Delta \quad B, \Gamma \Rightarrow \Delta}{A \vee B, \Gamma \Rightarrow \Delta}\ \mathsf{L}\vee \qquad \frac{\Gamma \Rightarrow \Delta, A}{\Gamma \Rightarrow \Delta, A \vee B}\ \mathsf{R}\vee_1 \qquad \frac{\Gamma \Rightarrow \Delta, B}{\Gamma \Rightarrow \Delta, A \vee B}\ \mathsf{R}\vee_2$$

$$\frac{\Gamma \Rightarrow \Delta, A}{\neg A, \Gamma \Rightarrow \Delta}\ \mathsf{L}\neg \qquad \frac{A, \Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta, \neg A}\ \mathsf{R}\neg$$

$$\frac{\Gamma \Rightarrow \Delta, A \quad B, \Gamma' \Rightarrow \Delta'}{A \to B, \Gamma, \Gamma' \Rightarrow \Delta, \Delta'}\ \mathsf{L}\to \qquad \frac{A, \Gamma \Rightarrow \Delta, B}{\Gamma \Rightarrow \Delta, A \to B}\ \mathsf{R}\to$$

Note that the rules for $\to$ can be derived using the definition $A \to B =_{def} \neg A \oplus B$. (Rules for additive implication are omitted but can be derived similarly.)

Figure 1: Rules for **GMALL**.

$$\frac{A, \Gamma \Rightarrow \Delta \quad A, \Gamma' \Rightarrow \Delta'}{\diamond A, \Gamma, \Gamma' \Rightarrow \Delta, \Delta'}\ \mathsf{L}\diamond \qquad \frac{\Gamma \Rightarrow \Delta, A, A}{\Gamma \Rightarrow \Delta, \diamond A}\ \mathsf{R}\diamond$$

$$\frac{A, A, \Gamma \Rightarrow \Delta}{\boxdot A, \Gamma \Rightarrow \Delta}\ \mathsf{L}\boxdot \qquad \frac{\Gamma \Rightarrow \Delta, A \quad \Gamma' \Rightarrow \Delta', A}{\Gamma, \Gamma' \Rightarrow \Delta, \Delta', \boxdot A}\ \mathsf{R}\boxdot$$

Figure 2: Derived rules for Tarskian modalities in **GMALL**.

Remark 1. Because of the symmetries that occur in many of the proofs given in this paper, the following non-branching forms of the modal rules will be used for brevity:

$$\frac{A, \Gamma \Rightarrow \Delta}{\diamond A, \Gamma, \Gamma \Rightarrow \Delta, \Delta}\ \mathsf{L}\diamond \qquad \frac{\Gamma \Rightarrow \Delta, A}{\Gamma, \Gamma \Rightarrow \Delta, \Delta, \boxdot A}\ \mathsf{R}\boxdot$$

Proposition 1. *The following equivalences hold in* **MALL***:*

$$\diamond A \equiv A \oplus A \tag{3}$$

$$\boxdot A \equiv A \otimes A \tag{4}$$

Proof. Straightforward. $\square$

Remark 2. The equivalences in Proposition 1 may be used as alternative definitions of the Tarskian modalities.

Proposition 2. *The following are derivable in* **GMALL***:*

$$\neg \boxdot \bot \tag{5}$$
$$\diamonddot \top \tag{6}$$
$$\neg \boxdot (A \wedge \neg A) \tag{7}$$
$$\diamonddot (A \vee \neg A) \tag{8}$$

where (6) *corresponds to the intuitionistic modal axiom* D [18].

Proof. Straightforward. □

Proposition 3. *The* $\mathsf{K}\boxdot$ *rule and its dual*

$$\frac{\Gamma \Rightarrow A}{\boxdot\Gamma \Rightarrow \boxdot A}\ \mathsf{K}\boxdot \qquad \frac{A \Rightarrow \Delta}{\diamonddot A \Rightarrow \diamonddot \Delta}\ \mathsf{K}\diamonddot$$

are derivable in **GMALL***.*

Proof. Straightforward. □

Proposition 4 (Distribution Theorems)**.** *The following are derivable in* **GMALL***:*

$$\boxdot(A \to B) \to (\boxdot A \to \boxdot B) \tag{9}$$
$$\boxdot(A \wedge B) \to (\boxdot A \wedge \boxdot B) \tag{10}$$
$$\diamonddot(A \wedge B) \to (\diamonddot A \wedge \diamonddot B) \tag{11}$$
$$(\boxdot A \vee \boxdot B) \to \boxdot(A \vee B) \tag{12}$$
$$(\diamonddot A \vee \diamonddot B) \to \diamonddot(A \vee B) \tag{13}$$

where (9) *corresponds to the* K *axiom.*

Proof. Straightforward. □

3. A Comparison of **MALL** with Substructural-**K** (**KMALL**)

Definition 1 (Substructural-**K**)**.** Let Substructural-**K** (**KMALL**) be **MALL** augmented by extending the language of formulae with $\Box A$. We obtain a calculus **GKMALL** for **KMALL** by adding the following rule to **GMALL**:

$$\frac{\Gamma \Rightarrow A}{\Box\Gamma \Rightarrow \Box A}\ \mathsf{K}\Box$$

which corresponds to adding to **MALL** the necessity rule

$$\frac{\vdash A}{\vdash \Box A}\ \mathsf{N}$$

and the K axiom, $\Box(A \to B) \to (\Box A \to \Box B)$.

Remark 3. Cut elimination for **GKMALL** is shown in Appendix A.

Theorem 5. *If* **GKMALL** $\vdash \Gamma \Rightarrow \Delta$, *then* **GMALL** $\vdash [\boxdot/\Box]\Gamma \Rightarrow [\boxdot/\Box]\Delta$.

Proof. By induction on the derivation height. Note that $\Box$-formulae are be introduced into a **KMALL** derivation either by axioms $\mathsf{L}\bot$ and $\mathsf{R}\top$, or by the $\mathsf{K}\Box$ rule. Instances of $\mathsf{K}\Box$ are replaced by instances of $\mathsf{K}\boxdot$. □

Remark 4. **GKMALL** $\nvdash \neg\Box\bot, \Diamond\top, \neg\Box(A \wedge \neg A)$ and $\Diamond(A \vee \neg A)$. A form of the $\mathsf{K}\Box$ rule that allows for empty succedents, e.g.

$$\frac{\Gamma \Rightarrow \Delta}{\Box\Gamma \Rightarrow \Box\Delta}\ \mathsf{K}\Box'$$

where $|\Delta| \leq 1$, would allow for the derivation of $\neg\Box\bot$ (5) and $\Diamond\top$ (6) but not $\neg\Box(A \wedge \neg A)$ (7) and $\Diamond(A \vee \neg A)$ (8).

Remark 5. The converse of (10), $(\boxdot A \wedge \boxdot B) \to \boxdot(A \wedge B)$ is not derivable in either **GMALL** or **GKMALL**.

4. Multiplicative Additive Linear Logic with Mingle

Linear Logics with mingle are discussed in [10, 11]. The mingle (also called "merge" or "mix") rule is:

$$\frac{\Gamma \Rightarrow \Delta \quad \Gamma' \Rightarrow \Delta'}{\Gamma, \Gamma' \Rightarrow \Delta, \Delta'}\ \mathsf{M}$$

Proposition 6. *The following are derivable in* **GMALL** $+\ \mathsf{M}$*:*

$$\boxdot A \to \Diamonddot A \tag{14}$$

$$(\Diamonddot A \to \Diamonddot B) \to \Diamonddot(A \to B) \tag{15}$$

where (14) *corresponds to a form of the* D *axiom.*

Remark 6. We note that the formulae (14) and (15) can be derived using anti-contraction (duplication) rules as well:

$$\frac{A, \Gamma \Rightarrow \Delta}{A, A, \Gamma \Rightarrow \Delta}\ \mathsf{LC}^{-1} \qquad \frac{\Gamma \Rightarrow \Delta, A}{\Gamma \Rightarrow \Delta, A, A}\ \mathsf{RC}^{-1}$$

5. Affine Logic

Affine Logic [7], also called Affine Multiplicative Additive Linear Logic (**AMALL**), is **MALL** augmented with the weakening axiom, $(A \to 1) \wedge (0 \to A)$. The corresponding calculus, **GAMALL** is obtained by adding weakening rules to **GMALL**:

$$\frac{\Gamma \Rightarrow \Delta}{A, \Gamma \Rightarrow \Delta}\ \mathsf{LW} \qquad \frac{\Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta, A}\ \mathsf{RW}$$

Remark 7. In Affine Logic, $0 \equiv \bot$ and $1 \equiv \top$.

Proposition 7. *If* **GMALL** $+\ \mathsf{M} \vdash A$*, then* **GAMALL** $\vdash A$*.*

Proof. M is admissible in **GAMALL**. ◻

Remark 8. Hence, the formulae in Proposition 6 are derivable in **GAMALL**.

Proposition 8 (Additive Modal Rules)**.** *The following rules can be derived in* **GAMALL***:*

$$\frac{\Gamma \Rightarrow \Delta, A}{\Gamma \Rightarrow \Delta, \Diamonddot A}\ \mathsf{R}\Diamonddot' \qquad \frac{\Gamma, A \Rightarrow \Delta}{\Gamma, \boxdot A \Rightarrow \Delta}\ \mathsf{L}\boxdot'$$

Proof. Straightforward. ◻

Proposition 9. *The following are derivable in* **GAMALL***:*

$$\boxdot A \to A \tag{16}$$

$$A \to \Diamonddot A \tag{17}$$

where (16) *corresponds to the* T *axiom.*

Proof. Straightforward. □

Proposition 10. *The following rules are derivable in* **GAMALL***:*

$$\frac{A \Rightarrow B}{\Rightarrow \Diamond(A \to \boxdot B)}\ \mathsf{R\Diamond R\boxdot} \qquad \frac{A \Rightarrow B}{\Rightarrow \Diamond(\Diamond A \to B)}\ \mathsf{R\Diamond L\Diamond}$$

Proof. Straightforward. □

Proposition 11. *The following are derivable in* **GAMALL***:*

$$\Diamond(\boxdot A \to \boxdot \boxdot A) \tag{18}$$
$$\Diamond(\Diamond A \to \boxdot\Diamond A) \tag{19}$$
$$\Diamond(A \to \boxdot\Diamond A) \tag{20}$$
$$\Diamond(\Diamond A \to A) \tag{21}$$
$$\Diamond(A \to \boxdot A) \tag{22}$$
$$\Diamond(A \to \boxdot B) \to \Diamond(\boxdot A \to B) \tag{23}$$

where (18), (19) *and* (20) *are "*$\Diamond$*-forms" of the* S4, S5 *and* B *axioms, respectively.*

Proof. Straightforward. □

6. Strict Logic

A calculus **GSLL** for Strict Linear Logic (**SLL**) is obtained by adding to **GMALL** the contraction rules:

$$\frac{A, A, \Gamma \Rightarrow \Delta}{A, \Gamma \Rightarrow \Delta}\ \mathsf{LC} \qquad \frac{\Gamma \Rightarrow \Delta, A, A}{\Gamma \Rightarrow \Delta, A}\ \mathsf{RC}$$

Proposition 12. *The following can be derived in* **GSLL***:*

$$A \to \boxdot A \tag{24}$$
$$\Diamond A \to A \tag{25}$$
$$\boxdot A \to \boxdot \boxdot A \tag{26}$$
$$\Diamond\Diamond A \to A\Diamond \tag{27}$$
$$\Diamond A \to \boxdot\Diamond A \tag{28}$$
$$A \to \boxdot\Diamond A \tag{29}$$

where (26), (28) *and* (29) *correspond to the* S4, B *and* **S5** *axioms, respectively.*

Proof. Straightforward. □

Proposition 13. *The following can be derived in* **GSLL***:*

$$(\boxdot A \wedge \boxdot B) \to \boxdot(A \wedge B) \tag{30}$$
$$(\Diamond A \wedge \Diamond B) \to \Diamond(A \wedge B) \tag{31}$$
$$\boxdot(A \vee B) \to (\boxdot A \vee \boxdot B) \tag{32}$$
$$\Diamond(A \vee B) \to (\Diamond A \vee \Diamond B) \tag{33}$$

Proof. Straightforward. □

Remark 9. These are the converse of formulae (30) through (33). Note that (31) is the same formulae as (2) mentioned in the introduction. Indeed (30) may also be considered paradoxical.

7. Involutive Uninorm Logic

Involutive Uninorm Logic (**IUL**) [15] is a substructural fuzzy logic, and has a hypersequent calculus **GIUL** (Figure 3) based on a hyperextension of **GMALL** [1] and the communication (Com) rule.

$$\frac{}{0 \Rightarrow}\ \mathsf{L0} \qquad \frac{\mathcal{H} \mid \Gamma \Rightarrow \Delta}{\mathcal{H} \mid \Gamma \Rightarrow \Delta, 0}\ \mathsf{R0} \qquad \frac{\mathcal{H} \mid \Gamma \Rightarrow \Delta}{\mathcal{H} \mid 1, \Gamma \Rightarrow \Delta}\ \mathsf{L1} \qquad \frac{}{\Rightarrow 1}\ \mathsf{R1}$$

$$\frac{}{\bot, \Gamma \Rightarrow \Delta}\ \mathsf{L}\bot \qquad \frac{}{\Gamma \Rightarrow \Delta, \top}\ \mathsf{R}\top$$

$$\frac{\mathcal{H} \mid \Gamma, A, B \Rightarrow \Delta}{\mathcal{H} \mid A \otimes B, \Gamma \Rightarrow \Delta}\ \mathsf{L}\otimes \qquad \frac{\mathcal{H} \mid \Gamma \Rightarrow \Delta, A \quad \mathcal{H} \mid \Gamma' \Rightarrow \Delta', B}{\mathcal{H} \mid \Gamma, \Gamma' \Rightarrow \Delta, \Delta', A \otimes B}\ \mathsf{R}\otimes$$

$$\frac{\mathcal{H} \mid A, \Gamma \Rightarrow \Delta \quad \mathcal{H} \mid B, \Gamma' \Rightarrow \Delta'}{\mathcal{H} \mid A \oplus B, \Gamma, \Gamma' \Rightarrow \Delta, \Delta'}\ \mathsf{L}\oplus \qquad \frac{\mathcal{H} \mid \Gamma \Rightarrow \Delta, A, B}{\mathcal{H} \mid \Gamma \Rightarrow \Delta, A \oplus B}\ \mathsf{R}\oplus$$

$$\frac{\mathcal{H} \mid A, \Gamma \Rightarrow \Delta}{\mathcal{H} \mid A \wedge B, \Gamma \Rightarrow \Delta}\ \mathsf{L}\wedge_1 \qquad \frac{\mathcal{H} \mid B, \Gamma \Rightarrow \Delta}{\mathcal{H} \mid A \wedge B, \Gamma \Rightarrow \Delta}\ \mathsf{L}\wedge_2 \qquad \frac{\mathcal{H} \mid \Gamma \Rightarrow \Delta, A \quad \mathcal{H} \mid \Gamma \Rightarrow \Delta, B}{\mathcal{H} \mid \Gamma \Rightarrow \Delta, A \wedge B}\ \mathsf{R}\wedge$$

$$\frac{\mathcal{H} \mid A, \Gamma \Rightarrow \Delta \quad \mathcal{H} \mid B, \Gamma \Rightarrow \Delta}{\mathcal{H} \mid A \vee B, \Gamma \Rightarrow \Delta}\ \mathsf{L}\vee \qquad \frac{\mathcal{H} \mid \Gamma \Rightarrow \Delta, A}{\mathcal{H} \mid \Gamma \Rightarrow \Delta, A \vee B}\ \mathsf{R}\vee_1 \qquad \frac{\mathcal{H} \mid \Gamma \Rightarrow \Delta, B}{\mathcal{H} \mid \Gamma \Rightarrow \Delta, A \vee B}\ \mathsf{R}\vee_2$$

$$\frac{\mathcal{H} \mid \Gamma \Rightarrow \Delta, A}{\mathcal{H} \mid \neg A, \Gamma \Rightarrow \Delta}\ \mathsf{L}\neg \qquad \frac{\mathcal{H} \mid A, \Gamma \Rightarrow \Delta}{\mathcal{H} \mid \Gamma \Rightarrow \Delta, \neg A}\ \mathsf{R}\neg$$

$$\frac{\mathcal{H} \mid \Gamma \Rightarrow \Delta, A \quad \mathcal{H} \mid B, \Gamma' \Rightarrow \Delta'}{\mathcal{H} \mid A \to B, \Gamma, \Gamma' \Rightarrow \Delta, \Delta'}\ \mathsf{L}\to \qquad \frac{\mathcal{H} \mid A, \Gamma \Rightarrow \Delta, B}{\mathcal{H} \mid \Gamma \Rightarrow \Delta, A \to B}\ \mathsf{R}\to$$

$$\frac{\mathcal{H}}{\mathcal{H} \mid \Gamma \Rightarrow \Delta}\ \mathsf{EW} \qquad \frac{\mathcal{H} \mid \Gamma \Rightarrow \Delta \mid \Gamma \Rightarrow \Delta}{\mathcal{H} \mid \Gamma \Rightarrow \Delta}\ \mathsf{EC}$$

$$\frac{\mathcal{H} \mid \Gamma_1, \Pi_1 \Rightarrow \Sigma_1, \Delta_1 \quad \mathcal{H} \mid \Gamma_2, \Pi_2 \Rightarrow \Sigma_2, \Delta_2}{\mathcal{H} \mid \Gamma_1, \Gamma_2 \Rightarrow \Delta_1, \Delta_2 \mid \Pi_1, \Pi_2 \Rightarrow \Sigma_1, \Sigma_2}\ \mathsf{Com}$$

Figure 3: Rules for **GIUL**.

Proposition 14. *The following rules are derivable in* **GIUL***:*

$$\frac{\mathcal{H} \mid A, \Gamma \Rightarrow \Delta \mid B, \Gamma \Rightarrow \Delta}{\mathcal{H} \mid A \wedge B, \Gamma \Rightarrow \Delta}\ \mathsf{L}\wedge' \qquad \frac{\mathcal{H} \mid \Gamma \Rightarrow \Delta, A \mid \Gamma \Rightarrow \Delta, B}{\mathcal{H} \mid \Gamma \Rightarrow \Delta, A \vee B}\ \mathsf{R}\vee'$$

Proof. Straightforward, using EC. ◻

Proposition 15. *Formulae* (30) *through* (33) *are derivable in* **GIUL**.

Proof. Straightforward, using rules from Proposition 14 and Com. A proof of (30):

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{A\Rightarrow A}{A\Rightarrow A \mid B\Rightarrow A\wedge B}\,\mathsf{EW} \qquad \dfrac{\dfrac{A\Rightarrow A \quad B\Rightarrow B}{A\Rightarrow B\mid B\Rightarrow A}\,\mathsf{Com}\quad \dfrac{B\Rightarrow B}{A\Rightarrow B\mid B\Rightarrow B}\,\mathsf{EW}}{A\Rightarrow B\mid B\Rightarrow A\wedge B}\,\mathsf{R}\wedge}{A\Rightarrow A\wedge B\mid B\Rightarrow A\wedge B}\,\mathsf{R}\wedge}{A\Rightarrow A\wedge B\mid B, B\Rightarrow \boxdot(A\wedge B)}\,\mathsf{R}\boxdot}{A, A\Rightarrow \boxdot(A\wedge B)\mid B, B\Rightarrow \boxdot(A\wedge B)}\,\mathsf{R}\boxdot}{\boxdot A\Rightarrow \boxdot(A\wedge B)\mid \boxdot B\Rightarrow \boxdot(A\wedge B)}\,\mathsf{L}\boxdot^2}{\boxdot A\wedge \boxdot B\Rightarrow \boxdot(A\wedge B)}\,\mathsf{L}\wedge'}{\Rightarrow (\boxdot A\wedge \boxdot B)\to \boxdot(A\wedge B)}\,\mathsf{R}\to$$

Proofs of (31) through (33) are similar.

□

8. Discussion and Future Work

Much of the content in this paper is straightforward. However, the formal properties of the Tarskian möglichkeit are of interest.

Theorem 5 is noteworthy, in that all of the derivable modal sequents derivable in **GKMALL** correspond to derivable modal sequents in **GMALL** using the Tarskian möglichkeit.

A semantic characterisation of the Tarskian modalities with respect to various logics is in process.

A deeper comparison of substructural logics with Tarskian modalities and their counterpart extensions to **KMALL** is an area of future investigation.

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Appendix A. Substructural-K (KMALL)

Lemma 16 (Cut Admissibility)**. GKMALL** *admits cut*

$$\dfrac{\Gamma\Rightarrow\Delta, A \quad A, \Gamma'\Rightarrow\Delta'}{\Gamma, \Gamma'\Rightarrow\Delta, \Delta'}\,\mathsf{cut}$$

Proof. Note that **GMALL** admits cut [19]. Adding $\mathsf{K}\Box$ to **GKMALL** also admits cut, by induction on the derivation height, with the following cases:

(1) If the cut formula is not of the form $\Box A$, then permute cut upwards.
(2) If the cut formula is not the principal formula on either premiss, permute the cut upward on that premiss.
(3) If the cut formula is the principal formula of either an instance of $\mathsf{L}\bot$ or $\mathsf{R}\top$, then so is the conclusion of the cut.
(4) The remaining case is that both premisses of the cut are the conclusions of instances of $\mathsf{K}\Box$. The cut is then permuted to the premisses of both $\mathsf{K}\Box$ instances, and $\mathsf{K}\Box$ is applied to the conclusion of the cut.

□

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